On representations of an operator algebra and the transfer matrix spectrum of the q-state Potts model

1986 J. Phys. A: Math. Gen. 19 L1117

(http://iopscience.iop.org/0305-4470/19/18/001)

J. Phys. A: Math. Gen. 19 (1986) L1117–L1123. Printed in Great Britain

LETTER TO THE EDITOR

On representations of an operator algebra and the transfer matrix spectrum of the q-state Potts model

P P Martin
Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK

Received 1 August 1986

Abstract. We show that the Potts and Temperley–Lieb representations of the projection generators of a von Neumann algebra are, in general, reducible. We write down a new representation with non-vanishing product of odd generators R and give evidence that it is a common element in these reductions. The operators in this representation may be interpreted as giving the bond transfer matrices of the square lattice Whitney polynomial.

We show that the reduction of the Temperley–Lieb representation also contains irreducible representations with $R=0$ which are responsible for eigenvalues in the ice-model spectrum independent of those of the Potts model.

In this letter we report some recent work on the use of operators obeying algebraic relations associated with the projection generators of von Neumann algebras (Jones 1983) in relating the transfer matrix spectra of statistical mechanical models. In particular we explain the limitations of the relationship between the square lattice Potts and staggered ice-type models (compare Baxter (1982a) with Baxter (1982b) and Martin (1986)). The transfer matrices for these models are constructed from different representations of the same algebra (Temperley and Lieb 1971, Baxter 1982a). We review this construction and then show that both representations are reducible and have some, but not all, component irreducible representations in common. We show in general that the eigenvalues of representations of operator products, such as these transfer matrices, are only determined by the algebraic relations when the product of odd generators R is non-vanishing in an irreducible representation. We show in particular that $R=0$ representations give rise to different eigenvalues in Potts and ice-type models. We then write down a new $R\neq 0$ representation which generates the transfer matrix for the square lattice Whitney polynomial (or dichromatic polynomial (Baxter 1982a, Blöte and Nightingale 1982)) and give evidence that it is equivalent to the irreducible $R\neq 0$ element in the Potts and Temperley–Lieb representations.

The n-site layer transfer matrix for the square lattice q-state Potts model may be written

$$T=\left(\prod_{j=1}^{n}(v+q^{1/2}U_{2j-1})\right)\left(\prod_{j=1}^{n-1}(1+q^{-1/2}vU_{2j})\right) \tag{1}$$

where $v=\exp(\beta)-1$ (Baxter 1982a).

The matrices $\{U_i,\ i=1,2n-1\}$ are given by

$$U_{2i-1}=I\times I\times\ldots\times A\times\ldots\times I \tag{2}$$

(where $I_{jk} = \delta_{jk}$ and $A_{jk} = q^{-1/2}$, with $j, k = 1, \ldots, q$ and A appearing in the ith position in the cross product) and similarly

$$U_{2i} = q^{-1/2} \sum_{r=0}^{q-1} B_i(r) B_{i+1}(r)^{\dagger} \tag{3}$$

where

$$B_i(r) = I \times I \times \ldots \times C(r) \times \ldots \times I$$

and

$$(C(r))_{jk} = \delta_{jk} \exp\left(\frac{2\pi i r(j-1)}{q}\right).$$

These matrices satisfy the following relations (for projection generators of a von Neumann algebra—see, for example, Jones (1983); see also Temperley (1986)):

$$U_i U_i = q^{1/2} U_i$$

$$U_i U_{i\pm1} U_i = U_i$$

$$U_i U_j = U_j U_i \qquad |i-j| \geq 2. \tag{4}$$

An alternative set of matrices satisfying these relations, given by Temperley and Lieb (1971) (see also Baxter 1982a), may be written as

$$U_{2i-1} = I_4 \times I_4 \times \ldots \times U \times \ldots \times I_4 \tag{5}$$

$$U_{2i} = I_4 \times I_4 \times \ldots \times I_2 \times U \times I_2 \times \ldots \times I_4 \tag{6}$$

or equivalently

$$U_i = I_2 \times I_2 \times \ldots \times U \times \ldots \times I_2$$

where

$$U = \begin{pmatrix} 0 & & & \\ & s^4 & 1 & \\ & 1 & s^{-4} & \\ & & & 0 \end{pmatrix}$$

appears in the ith position, I_m is the $m \times m$ unit matrix and $s^4 + s^{-4} = q^{1/2}$ (we make no notational distinction between representations). If these matrices are used in equation (1) we obtain the transfer matrix for the medial lattice staggered ice model.

It is easy to see that the two versions have some eigenvalues in common. The scalar $\tau(\chi)$ in

$$R\chi R = \tau(\chi) R \tag{7}$$

(where $R = \Pi_{i=1,n} U_{2i-1}$ and χ is any sum of products of U and 1) is determined by the von Neumann relations (4) (see Baxter 1982a). However, we now know that the equivalence between the ice and Potts models does not extend to all eigenvalues (it does not necessarily include the largest eigenvalue, for instance, see Baxter (1982b) and Martin (1986)). To understand the extent of the equivalence we must examine representation structure of operators satisfying the relations (4). In particular, we must decompose the known representations into their irreducible parts and classify these in terms of R, since when $R = 0$ equation (7) provides no constraint on the spectrum of χ.

The reducibility of the Potts representation, for example, is conveniently exemplified in the $q = 3$ case. Consider the change of basis effected by the similarity transformation matrix:

$$\mathcal{S} = S \times S \times \ldots \times S \tag{8}$$

where

$$S = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 \end{pmatrix}.$$

The representation becomes

$$U_{2i-1} = I_3 \times I_3 \times \ldots \times \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \times \ldots \times I_3$$

and

$$U_{2i} = I_3 \times I_3 \times \ldots \times D \times \ldots \times I_3 \tag{9}$$

where

$$D = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & \frac{1}{18} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{9} & 0 & 0 & 0 & \frac{1}{12} \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{9} & 0 & 0 & 0 & \frac{1}{12} \\ \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} & 0 \\ 2 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{9} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Because the block diagonal structure of D is invariant under cyclic permutation of elements in the cross product the representation decomposes. Note that only one component has $R \neq 0$ and that there can only be one irreducible component with $R \neq 0$. The $R \neq 0$ component at this stage has dimension d_n, where $d_1 = 2$ and $d_n = 3d_{n-1} - 1$. However, this component is itself reducible. The required basis cannot be expressed simply in terms of cross products, but the final reduced dimension $\bar{d}_n$ of the $R \neq 0$ component can be found as follows. By cyclically permutating the order of elements in the cross product (9) a U_{2n} and U_{2n+1} can be obtained from the d_n-dimensional matrices. This provides the representation for $n \to n+1$ (see (4)), so that $\bar{d}_n = d_{n-1}$ (equivalently for $q = 2$, of course, the reduced dimension $\bar{e}_n = 2\bar{e}_{n-1}$ with $\bar{e} = 1$).

The decomposition of the Temperley-Lieb representation is not so straighforward. We can, however, make an initial reduction into a direct sum of $2n+1$ representations of dimension $(2n)!/m!(2n-m)!$ $(m = 0, \ldots, 2n)$. This can be seen by considering a basis in which, schematically, entries in the template matrix

$$M = F \times F \times \ldots \times F \tag{10}$$

where

$$F = \begin{pmatrix} F_{+1} & F_0 \\ F_0 & F_{-1} \end{pmatrix}$$

are collected into blocks, each of whose elements have the same aggregate subscript. In ice-model notation this is the basis in which fixed numbers of up (down) arrows are grouped together (see Baxter 1982a).

Writing the odd operators in the form

$$U_{2i-1} = E_{i1} \times E_{i2} \times \ldots \times E_{in} \tag{11}$$

where

$$E_{ij} = E_{ij}^{+} \oplus E_{ij}^{0} \oplus E_{ij}^{-}$$

with

$$E_{ij}^{\pm} = 1 - \delta_{ij}$$

and

$$E_{ij}^{0} = I_2 + \delta_{ij} \begin{pmatrix} s^4 - 1 & 1 \\ 1 & s^{-4} - 1 \end{pmatrix}$$

we then see that the only representation with $R \neq 0$ is that of dimension $(2n)!/(n!)^2$ (with zero aggregate subscript in the above scheme, i.e. corresponding to the direct sum of terms in the expansion of (11) with equal numbers of E^+ and E^-). This representation is further reducible, but the appropriate basis is rather complicated in general. We can best illustrate the situation with some finite lattice examples.

Consider the smallest non-trivial case, $n = 2$. Here we have (for $q \neq 2$) a basis in which

$$U_1 = q^{1/2} \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

$$U_2 = q^{-1/2} \begin{pmatrix} 1 & q-1 & & & & \\ 1 & q-1 & & & & \\ & & 1 & 1 & q-2 & \\ & & 1 & 1 & q-2 & \\ & & 1 & 1 & q-2 & \\ & & & & & 0 \end{pmatrix} \tag{12}$$

$$U_3 = q^{1/2} \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}.$$

It is easy to see, for example from (9), that in this case all three irreducible representations are also found in the decomposition of the Potts representation. This is not true in general. For example, at $q=2$ the similarity transformation used above becomes singular, and instead we find (with U_1 and U_3 as above)

$$U_2 = q^{-1/2}\begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The first two 2×2 blocks on the diagonal in fact give two irreducible representations. These also appear in the Potts representation, thus giving, in this case, the full Potts-model spectrum (clearly, equivalent representations have the same spectrum irrespective of the value of R).

The remaining elements only give a representation when taken in conjunction with the second block (although they do not interfere with the algebra of this block). This 'parasitic' representation does not appear in the decomposition of the Potts representation and is responsible for independent eigenvalues in the ice-model spectrum. It also precludes the existence of a basis in which we simply have a direct sum of irreducible representations (cf group representation theory).

The situation for $n=3$ is similar but algebraically more complicated. However, having already shown that inequivalent representations with independent spectra can and do appear in the two models, we should now consider the extent of common ground between them. To this end, as we will see, it is not necessary to consider individual cases. It is possible to check from (11) that the Temperley-Lieb representation, like the Potts representation, contains only one irreducible representation with $R\neq 0$. We wiil show below that all irreducible representations (μ) with $R_\mu \neq 0$ have the same dimension and give the same spectrum for a given χ (defined as in (7)). We will then show how to construct such representations much more directly.

Firstly note that any χ_μ may be written $\chi_\mu(\varepsilon)|_{\varepsilon=0}$, where the coefficients of the characteristic polynomial depend continuously on ε and where, for any finite ε, we have a $\chi_\mu(\varepsilon)$ with no symmetries, i.e. whose eigenvalues are all different roots on the same Riemann surface (Phillips 1957). All the eigenvalues of $\chi_\mu(\varepsilon)$ are then determined from a knowledge of one. Now since $R\neq 0$ one eigenvalue is determined independently of μ, thus all are determined and by continuity all are determined for χ_μ.

Now consider the following construction. Labelling the rows and columns of matrices by the possible connectivities (Blöte and Nightingale 1982) of n sites then

$(U_{2i-1})_{jk} = q^{1/2}$ if disconnecting the ith site changes the connectivity from j to k

$= 0$ otherwise

and

$(U_{2i})_{jk} = q^{\delta_{jk}-1/2}$ if connecting the ith and $(i+1)$th sites changes the connectivity from j to k

$= 0$ otherwise.

It is easy to see that these matrices obey the relations (4). The representation dimension C_n is given by the number of connectivities

$$C_1 = 1$$

$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

(Blöte and Nightingale 1982), and clearly $R \neq 0$. If this representation is used in (1) we obtain the n-site layer transfer matrix for the square lattice Whitney polynomial, with each individual factor an appropriate bond transfer matrix.

As far as we have checked (up to $n = 5$) this representation is irreducible for general q, except at $q = 1, 2, 3$ where the $R \neq 0$ component is equivalent to that contained in the reduction of the Potts representation (although, as in (13), the decomposition cannot be written as a direct sum). In the $q = 3$ case the new representation only becomes reducible at $n = 5$, corresponding to the divergence of $\bar{d}_n$ from C_n at that point.

In fact, these observations are consistent with Temperley's (1986) Young tableaux construction. The dimensionality C_n is equal to the (q-independent) number of allowed tableaux with two equal rows (compare Blöte and Nightingale (1982) with Temperley (1986): $(2n)!/(n!)^2$ also gives the total number of allowed tableaux). But is its possible to see that the construction breaks down for the corresponding representation at $n+1 = \pi/\cos^{-1}(\sqrt{q}/2)$ $(0 < q \leq 4)$ and for other representations at $n = \pi/\cos^{-1}(\sqrt{q}/2)$. This picks out $q = 1, 2, 3$ as special cases at precisely the right n values. Of course, as n increases it also picks out the other (non-integer) Beraha numbers (Baxter 1982a), but as these have no Potts model interpretation their significance is less clear. It is presumably *de rigueur* to conjecture that these models at criticality are part of the conformal series (see, for example, Friedan *et al* 1984). Certainly the relations (4) are closely related to the algebra used by Schultz *et al* (1964) when $q = 2$. We will discuss this aspect elsewhere.

The asymptotic behaviour for large n of the dimensions of these representations may be summarised as follows:

q-state Potts	$= q^n$
two-state Potts (reduced)	$\bar{e}_n = 2^{n-1}$
three-state Potts (reduced)	$\bar{d}_n \sim 3^{n-1-\ln 2/\ln 3}$
Temperley-Lieb	$= 4^n$
	$\frac{(2n)!}{(n!)^2} \sim 4^{n-(1/2\ln 4)\ln n+O(1)}$
Whitney	$C_n \sim 4^{n-(3/2\ln 4)\ln n+O(1)}$
	$= q^{(1-(q-4)/4\ln 4)n+O(\ln n)}$
	(for $q \approx 4$).

The latter result is yet another example of the special nature of $q = 4$ when q is thought of as a continuous variable (Baxter 1982a).

Note that the reduced two- and three-state cases give, in the limit, the internal symmetry factors we expect. Spatial symmetries only appear at the level of the full transfer matrix, i.e. when all sites are summed over (see Schultz *et al* 1964, Martin 1986). We observe also that, although $R \neq 0$ representations must give rise to the same spectrum even if inequivalent, the ones we have found are in fact equivalent.

Finally, note that R may be interpreted as corresponding to a particular boundary condition (see Baxter 1982a) in a particular representation and, although operator products corresponding to different boundary conditions may be constructed, R is the only one for which equations of the form of (7) (characterised by a purely scalar dependence on χ on the right-hand side) can be formed. In this sense any coincidence of transfer matrix eigenvalues between Potts and ice models outside the $R \neq 0$ irreducible subspace (for example, due to the occurrence of equivalent $R = 0$ representations) is accidental.

We have found that the relations (4) obeyed by matrices used to construct the Potts and ice-model transfer matrices only define their spectra subject to a further condition ($R_\mu \neq 0$) which is satisfied for part but not all of the spectrum in each case. Elsewhere the possibility of independent eigenvalues exists and is realised. We have given evidence that the further condition is, in general, satisfied by a new set of matrices which generate the transfer matrix for the Whitney polynomial. Our next step is to generalise the algebra for periodic boundary conditions, where exact finite lattice calculations (Martin 1986) indicate that the overlap of eigenvalues between Potts and ice models is even more restricted.

References

Baxter R J 1982a *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

—— 1982b *Proc. R. Soc.* A **383** 43

Blöte H W J and Nightingale M P 1982 *Physica* **112A** 405

Friedan D, Qiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575

Jones V F R 1983 *Invent. Math.* **72** 1

Martin P P 1986 *J. Phys. A: Math. Gen.* **19** 3267

Phillips E G 1957 *Functions of a Complex Variable* (Edinburgh: Oliver and Boyd) p 26

Schultz T D, Mattis D C and Lieb E H 1964 *Rev. Mod. Phys.* **36** 856

Temperley H N V 1986 *Potts Model and Related Problems. Private preprint*

Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251